The Role of Philosophy in Axiom Choice

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1 Introduction

In a recent paper in *Erkenntis* Kenny Easwaran proposes that a) the purpose of axioms is to help mathematicians avoid vain philosophical debates by articulating some principles that are obvious to everyone but b) if and when questions of mathematical interest turn out to be independent of the axioms currently needed, we will need to do philosophy to justify accepting new axioms. Thus, we get a picture of mathematics as working out what follows from a body of axioms currently accepted as obvious, and philosophy discovering which new propositions can be added as axioms. I emphatically agree with Easwaran about the usefulness of taking generally accepted claims as axioms, and trying to prove everything from them. But, I will argue, he's wrong about the claim that justifying new axioms requires doing philosophy (in any recognizable sense), rather than just more matheamtics.

Obviously, the boundary between mathematics and philosophy are not super sharp. But Easwaran suggests that justifying new axioms will require something like "refuting existing philosophical positions or showing that these positions support the new axiom", and gives the example of philosophical arguments against intuitionsim being used to support the axiom of choice, so he seems to be understanding "philosophy" in a pretty conventional way. I'm going to a) show how we might learn about new axioms in a way that has everything to do with conventional mathematics, and nothing to do with conventional philosophy and b) give some reasons for being skeptical that conventional philosophy (including philosophy of mathematics) could do much to justify axiom choice.

2 Mathematical Justification for New Axioms

So, here's how math could give us new axioms.

Mathematicians commonly come up with new kinds of mathematical objects (the objects and arrows of category theory are a recent example) and then propose axioms that seem obviously true of this new subject matter. Kenny

mentions this fact, and accepts that we can be justified in accepting these "subject matter specifying" axioms about a new domain of objects. Mathematicians also commonly think facts about one realm of objects can help us learn about some other domain (for example, facts about geometry are used to prove claims about arithmetic and vice versa). Now, suppose someone dreams up the Quons, some new kind of mathematical objects, and they state various axioms about the Quons which seem obvious to everyone who has seen even a brief motivation of what the Quons are supposed to be like, and then these axioms turn out to allow us to prove claims about the numbers which cannot be proved using the axioms currently accepted (i.e. the ZFC axioms).

Surely this is a mathematical process, not a philosophical one. The proponents of Quons are doing exactly the same thing as people who come up with any other kind of mathematical structure, and then apply reasoning about one domain to prove things about another domain. It just happens that the things which are obvious truths about what Quons would have to be like, and how facts about the Quons would fit together with facts about the numbers, in a way that allows us to prove new things. Surely that doesn't make what they are doing philosophy, in any intuitive sense of the word! And presumably, reasoning like this could justify us in accepting these new axioms about the Quons, just as people were justified in accepting, say, imaginary numbers and proofs of claims about the integers made using imaginary numbers.

Even more simply: mathematicians don't actually give explicit proofs of things from the ZFC arguments. They reason at a much higher level, and it turns out that most of what they accept as intuitively a good argument can be captured by first order logic + ZFC. (Indeed, the axioms of ZFC were cooked up exactly to have this property). So it's perfectly conceivable that there might turn out to be some perfectly intuitively obvious statement S or kind of mathematical argument, which could not be formalized in ZFC, but which felt as obvious as the least number principle when you considered it. In this case, again (barring the discovery of any unexpected paradoxical consequences or the like) we would presumably be justified in adding new axioms to ZFC to capture this reasoning.

Thus, it is easy to imagine how perfectly ordinary mathematical reasoning could (in principle) justify us in accepting new axioms.

3 Philosophical Justification for New Axioms

In contrast, it is not easy to see how philosophical reasoning could justify us in accepting new axioms.

Firstly, it's not at all clear to me that the defeat of intuitionism (to the extent that it was defeated) was what provided justification for the axiom of choice. Surely Easwaran doesn't mean to suggest that our justification for accepting AC depends on an argument like this: "Intuitionism entails that P,intuitionism is false, therefore P". So, presumably he thinks that we had some antecedent

justification for accepting AC, which arguments from intuitionism appeared to threaten. But what was the nature this justification? Does some opposing philosophical position, say Platonism, support the axiom of choice? It doesn't seem to. Platonism tells you that there are sets, and there's a fact of the matter about whether there's a choice function for any collection of non-empty sets. But this doesn't tell us anything about which way the fact of the matter goes!

Rather, it seems to be directly mathematically plausible that AC is true, in much the same way as the least number principle, or various facts about set theory that can be proved from ZF alone seem plausible. Thus, it would seem that even in the case Easwaran chose, we have mainly a mathematical justification which philosophical considerations merely threaten to defeat. But actually even then, one might think that controversy over AC arose more from its unintuitive mathematical consequences, like the paradoxical decomposition of the sphere, rather than intuitionism. For, when we apply Hume's method of difference by considering other mathematical claims which the intuitionists were equally worried about, but which did not lead to mathematically implausible looking consequences, we see that the mathematical community was much less troubled in ignoring worries about these. [Check this with Koellner]

And this brings us to a more general issue, analogous (ironically?) to the one pointed out by Frege with regard to tracing the justification for mathematical claims to psychology. The problem is this. In general, we're much more certain about mathematics (when we have a proof) than about almost anything in philosophy (hence the expression "mathematically certain". So: what kind of philosophical argument could we produce that would secure the degree of justification which we expect from a mathematical proof? Easwaran emphasizes how the practical value of mathematical arguments arises from their being accepted as obvious by everyone. But when has philosophy ever had this result? How could the justification of propositions which we take to be the most strongly justified (mathematical axioms), depend on something which generally provides such weak and tentative justifications as philosophy. Thus, to say that mathematics can tell us when a potential new axioms would have interesting consequences, but we need philosophy (not mathematics) to justify accepting it, seems like saying that dynamite can get you into the bank-vault with the safe, but you need a beach trowel to open it.

Thus, so far as I can tell, philosophy only figures in the justification of new mathematical axioms as something like a defeater-defeater in cases where there is some philosophical argument for doubting the axiom. And, further, at least in my experience as a young philosopher of mathematics, these cases are quite rare. This is because (unsurprisingly) the mathematical upshots of different philosophical positions tend to be quite general. There are boatloads of philosophical arguments about what sets are like, and cartloads of arguments about whether there can be any fact of the matter about questions which are independent of whatever mathematical axioms are accepted in a given community. But there are none concerning, say, the size of the contiuum. Even if we imag-

ine the most extreme possible victory for some position in the philosophy of mathematics, I can see nothing that would justify accepting CH, or CH. If the Formalist wins, we might learn there's no fact of the matter. But to learn that the Platonist is right that there are sets, tells us nothing about whether there's a set intermediate in size between the integers and the reals. Similarly, to learn that the Fictionalist is right, and there are no sets, but mathematics is all about what's true in the second-order fiction that there are sets, is not to tell us what the size of the continuum is in this fiction. Intuitionism seems to be the best case, but as noted above, it leads to fairly extreme doubt in the realm of mathematics, and it's hard to envisage being more confident in, say, Dummett's manifestation argument than in the claim that there are or are not infinitely many twin primes.

Thus, I am perplexed by how Easwaran imagines philosophy could justify new axioms for mathematics. At most, the victory of some form of antirealsim might tell us that there are no facts of the matter so it doesn't matter which new axioms we pick - but this is presumably not the kind of justification he has in mind.

4 Easwaran vs. Maddy

The arguments above may seem too easy, so let me conclude by addressing Easwaran's arguments against Maddy's version of the claim that axiom choice is mostly matter for mathematicians, and by trying to diagnose why his picture looks so attractive.

Maddy claims (at least as Easwaran reads her) that questions about axioms choice are mathematical questions, in the sense that the property of being the right axiom to choose is itself a mathematical one. The right axiom is the one that maximizes expressive power of the resultant thoery, unifies existing axioms etc. Easwaran then raises doubts that questions about the applications of these claims can be made mathematical.

Here, it suffices to emphasize that I am claiming that questions of axiom choice are mathematical rather than philosophical ones (to the extent that distinction makes any sense) in a quite different way from Maddy. My claim is that the kind of processes that might lead us to accept, say, axioms positing large cardinals, are the same as ones which might lead us to accept, reject, or remain unconvinced with regard to any other mathematical claim - namely, informal mathematical reasoning. No detour through the advocacy or attack of "philosophical positions" is required. The fact that these questions about large cardinals have been shown to be independent merely tells us that - if someone ever does come up with an intuitively compelling argument for the existence of large cardinals- this will not be formalizable in ZFC. They don't show that some kind of appeal to Kantian phenomenology or theories of meaning or meta-ontology are required. Indeed, if it turned out the only possible argument that

settled the question of whether there are large cardinals needed to appeal to such relatively uncertain objects of philosophy, commonsensical mathematicans could plausibly say this showed that we would never know whether there are large cardinals.

Thus, even assuming that Easwaran is right in his claim against Maddy, that it's impossible to phrase a mathematical criterion for which new axioms to accept, this poses no problem for my view, that the reasoning one engages in when deciding whether to e.g. accept large cardinal axioms (namely: trying to figure out whether there are large cardinals) requires nothing more than (informal) mathematical reasoning which gets applied to any other mathematical question.

5 On the Usefulness of Axioms

I think Easwaran's idea that we need philosophy to justify axiom choice seems immediately appealing for two reasons. Firstly, Easwaran rightly points out the use of having a commonly accepted axiomatic framework. The fact that mathematicians have a powerful body of propositions (the ZFC axioms) which are accepted as obvious by nearly everyone, and formalize nearly all intuitively good reasoning, provides the appealing possibility of breaking down one's arguments into steps which everyone is sure to accept. Thus, you might think that reasoning which cannot be justified in this way (as reasoning about what axioms to add obviously can't) has a very different character, and should more properly be called philosophy than mathematics.

But this idea is wrong, whether we understand it as a proposal about the extension of the word "mathematics" or as a psychological claim about why we should expect insights about which new axioms to accept to come from philosophy departments rather than mathematics departments.

Let's start with the conceptual analysis. The essence of mathematics, you might be tempted think, is proving things from generally accepted axioms, so any reasoning about which axioms to add, must involve something more than this - and hence cannot count as mathematics. But, one cannot say that reasoning which does not aim to work out consequences of any formal system isn't mathematics, on pain of saying that hundreds of years of number theory prior to Russell weren't mathematics. And you can't say that mathematics is reasoning which *can be justified in terms of ZFC*, since this would mean that any proofs constructed using the newly added axioms would not count as mathematics.

Now what about the psychological version of the claim? If we imagined that working mathematicians really explicitly proved things by consulting the ZFC axioms and deploying first order logic, there might be some plausibility to thinking that they wouldn't be able to tell us anything about what new axioms to add. But even in this case, we should note that a mathematican's knowledge and ability wouldn't be limited to their ability to follow the formal

proofs they created. For a human mathematician doesn't set out, like a machine, constructing every possible valid proof in ZFC+FOL! Instead, they have an ability to anticipate which approaches are likely to be fruitful, and what the answer to a given question is likely to be. The better they are at this skill, the less time the waste in fruitless approaches, and their expectations about how things "should" turn out, are constantly being corrected by experience of what proofs do actually get constructed.

Thus, in addition to whatever formal system captures the reasoning that a mathematician would accept as mathematically certain at a given time, there's a whole underground system of less-than-certain mathematical intuitions and heuristics, constantly being constantly being confirmed or corrected by experience. It's for this reason that I expect that if a) there is a fact of the matter about whether the continuum hypothesis is true or false and b) there is any route by which humanity could discover which one it is, this route will go through the informal reasoning of mathematicians (which is constantly battered and corrected at every moment of their professional lives) rather than through philosophers' arguments for and against general positions on the nature of mathematics.

6 On the Spookiness of Direct Mathematical Insight

The second reason why Easwaran's picture can seem (incorrectly, in my opinion) appealing, is the fact that we don't have any nice, generally accepted, naturalistic mechanism which accounts for the reliability of our mathematical judgements. This makes one hesitant to say that judgements of mathematical plausibility/obviousness could be immediately justified. For, doing so seems to require positing - and resting extreme trust in- a faculty of direct mathematical insight whose successful working seems to require co-operatation from the occult. (Contrast the case of visual perception, where we have a fairly well-understood naturalistic mechanism explaining vision, many people like the idea that we are directly justified in thinking there's a desk when we seem to see one.) That, in turn, leads to the thought that a) a mathematician's justification for believing something they find immediately obvious somehow "runs through" the ZFC axioms, and b) our acceptance of each of these propositions "runs through" some kind of hidden philosophical justification. If you put this all together, you get the conclusion that to justify adding a new axiom, what you would need is to point out a new/additional philosophical justification.

I have elsewhere proposed a naturalistic account of how we manage to be reliable about math, but even without that I think we can see problems in the little argument above. For, if you don't see how someone could be directly justified in judging a true mathematical proposition, (so that their justification really depends on a secret philosophical argument for this proposition), is it any

more clear how they could be directly justified in judging whatever premises are appealed to in the philosophical claim? We find ourselves confronting the well-known foundational regress problem in epistemology. But, whichever way you take out of the regress, it is extremely unclear how supplementing philosophical with mathematical justification could help of justification. If you are a foundationalist, why are indubitable mathematical propositions not just as good as indubitable philosophical ones (note that there's no generally accepted mechanism explaining our access to either of these subjects)? And if you are a coherentist, why think philosophical arguments in particular need to support the mathematically obvious, rather than other philosophical positions?

Lurking in the background here, perhaps we can detect the ghost of the logical positivist view, that mathematical statements could be traced back to logic plus statements that are analytic, and hence can be trivially justified by saying "that's just part of what I mean by the word". (Easwaran makes a distinction between axioms that can be said to be part of the concept of some mathematical notion vs. those that cannot, so perhaps one might hope that by drilling down to philosophical justifications for the remaining axioms one will arrive at a point where all premises involved can be justified as being 'part of some relevant concept'.) But remember how problematic this supposedly easy access to analytic truths turned out to be. If we take the notion of concept-inspection literally it would require exactly the kind of problematic visit to Frege's third realm which seems worrying about direct access to mathematical facts. On the other hand, if we understand this conceptual insight more concretely - as reporting an implicit definition, or a central aspect of how we use a word, we face a different problem. As the famous example of "tonk" showed, not all patterns of implicit definitions correspond to a genuine concept. But whether or not a proposed implicit definition is allows contradiction, or allows the derivation of every sentence or not etc. is a fact about provability mathematical fact. Thus (if one buys the idea of psychologically immediate judgements secretly depending on others for their justification) one's justification in asserting what you have derived from an attempted implicit definition D, will depend on a previous (implicit) mathematical judgement about D's being coherent/not allowing for the derivation of contradiction etc.

Thus (I claim) anxieties about how mathematical insight could work, ultimately provide little reason for thinking that mathematical axioms must be justified by an appeal to philosophy.

7 Conclusion

To sum up: In this essay I have argued that Kenny Easwaran is wrong to say that although mathematicians can delay philosophical questions by working with axioms that everyone accepts, they must eventually turn to philosophy to determine which axioms to add. Rather, if there is any hope of discovering new

true statements of number theory which are independent of ZFC, and hence can be added axioms, this comes from mathematicians' normal informal reasoning about the numbers- which is constantly checked, corrected and supported by experience- rather than from any philosophical claims about the nature of meaning or mathematical objects.

Let me emphasize in concluding, that in saying mathematicians don't need philosophy to justify accepting new axioms, I don't mean to say that there will always be a mathematical justification for adding an axiom to settle any question of interest (only that when we can justify accepting a new axiom, I would expect the justification to come from direct mathematics rather than consideration of the relationship between the proposed new axiom and philosophical theories). As a truth-value realist with regard to the philosophy of math, I accept a firm distinction between the truth or falsity of a mathematical statement and our ability to find a proof. Let a question about number theory be as beautiful and interesting as you like, this doesn't ensure that we can find proof or refutation. The sad facts of human psychology might be such that there's just way of deriving certain truths from any collection of other mathematical statements which feel obvious and compelling to human beings. We may even have already got all the axioms of set theory we could ever be justified in accepting in getting ZFC (plus a sequence of iterated consistency sentences, our confidence in each of which gradually decreases as one proceeds along the list), so that all statements which cannot be proved from this are forever inaccessible to us. If anything, I am grateful that the forces of evolution and Quinean theory revision have left us with intuitions about mathematics which can be combined in to yield the breadth and variety of knowledge which we have.