A Realist Solution to the Access Problem

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1 Introduction

Realists about mathematics face an apparent access problem[1]. We seem to have substantial mathematical knowledge. But, if mathematics really describes abstract, mind-independent objects, (as the realist contends) it seems mysterious how we could have gotten any such knowledge. Even the fact that we apparently have substantial *true beliefs* about mathematics can seem miraculous. The sets

[1]Major attempts to sharpen the worry have run into problems.

Benacerraf's version runs as follows: knowledge requires causal contact, and mathematical objects are a causal, hence we can't know things about them. But this depends on a causal constraint on knowledge, which is generally thought to be implausible (or at least, less plausible than the opposing arguments for realism about mathematical objects).

Hartry Field suggested that Platonists (presumably) believe that the following conditional reliably holds 'if mathematicians believe that P, then P', but they cannot explain why it holds. This gets a bit messy because maybe, for Davidsonain reasons, you don't count as believing or asserting anything (but only as making meaningless noises) at all if you're not largely reliable. So, there might be a trivial explanation for the fact that in all close possible worlds *where people count as having mathematical beliefs* they are largely reliable.

Oysten Linnebo's modification 'reliably: If mathematicians are inclined to say "S" then "S" expresses a truth' fixes this problem. For, close possible worlds (if there are any) where our mathematical practice is so badly off that the symbols we write on chalkboards are meaningless, are still ones where we make certain sounds, and hence still relevant to Linnebo's conditional.

But even this formulation can be given a trivial answer, if interpreted in a suitably uncharitable way. For, the access problem is, in essence, a problem about how we managed to get reliable methods for forming beliefs about mathematics. Given that we have reliable methods for forming true beliefs about math, it's not surprising that we have a substantial body of true beliefs. Given that you accept, say, the ZF axioms and first order logic, it's just a matter of mathematics that what you generate by applying logic to the ZF axioms will be a body of further true beliefs. What remains puzzling is to explain (without invoking a benevolent God to hardwire good methods of logical and mathematical reasoning into your brain) you could have got good belief forming methods like these, which accurately reflect facts about mathematical objects you have no causal contact with.

can't come down and kick me if I form false beliefs about them, so what can explain the match between them and my beliefs?

This intuitive worry, can be seen behind a lineage of 'transcendental arguments' with the following form: we have mathematical knowledge, it would be impossible (miraculous) that we had mathematical knowledge if P, therefore not P. Here are some of the most famous conclusions which philosophers have argued for in this way: our souls causally interacted with mathematical objects before we were born (Plato), mathematics concerns the most general form of sensible intuition (Kant), and mathematical statements are literally false (Hartry Field).

In this paper, I will propose a realist answer to the access problem, which accounts for our knowledge of mathematics without positing any causal contact with abstract objects, or spooky pre-established harmony. If this works, it blocks the above kind of transcendental arguments. There's no need to jump to radical conclusions about the literal falsehood of mathematics, or its limited validity, in order to account for our access to it.

2 Realism

Realism is infamously hard to define. Micheal Dummett [2] has famously proposed using acceptance of the law of the excluded middle as a criterion of realism. I won't commit to any such precise formulation. But, for what follows, let me stipulate that I take acceptance of the following three claims to be a sufficient condition for being a realist about mathematics.

- Mathematical objects literally exist.
- Mathematical statements can be true or false, and their truth or falsity does not depend on any contingent facts about human society.

[2]Dummett M., (1978) Truth and Other Enigmas (London: Duckworth)

- Someone can intelligibly wonder about some questions like 'are there infinitely many twin primes?' - even if they wouldn't accept any formalized method of reasoning which would allow them to derive either this sentence or its negation.

Note that, while this notion of realism is compatible with a 'robust' notion of what the existence of abstract objects amounts to, it is equally compatible with a thinner, 'merely logical', conception of existence. It's even compatible with the meta-ontological view that the above-mentioned distinction between thick and thin notions of objecthood, doesn't make sense.

2.1 Realism and indeterminacy

As realist about pigs, I think that pigs literally exist, and statements about them can be literally true or false, independent of any facts about human society. I can even meaningfully wonder about questions which are permanently beyond the reach of my observation, like whether there are pigs on the other side of the universe. However, if I consider Charles Travis' question of whether there's a pig on the doorstep, when there's a *roast* pig on the doorstep[3], I'm not committed to thinking this question has a determinate answer. I can be a realist about pigs, but think that our concept PIG is imprecise in certain ways.

Similarly, realism about mathematics does not require thinking there's a definite right answer to *all* questions involving numbers, sets, or other mathematical terms. For example, you might have an intuitive conception of a prime number as "something whose only divisors are one and itself" - and then be faced with the question of whether one is a prime. Later you precisify the definition, and conclude that it isn't.

More substantively, one might think that our concept of number is definite

[3]The Silence of the Senses, Mind, vol 113, n. 449, January 2004, pp. 59-94.

enough that there's a right answer to all questions in the language of arithmetic, but our concept of set is indeterminate with regard to how high up the hierarchy of sets is supposed to go -and hence there is no fact of the matter about whether it includes certain large cardinals. These two intuitions can seem to be in tension - the first requiring realism about mathematics, the second requiring antirealism. But, in fact, they are perfectly compatible, and typical of our realist attitude towards ordinary objects like pigs.

Indeed, there seems to be much *more* vagueness and indeterminacy with regard to ordinary physical objects than mathematical ones. So, it's just not true that being a realist about some subject matter, commits you to thinking that there are definite facts of the matter about *all* questions in that area.

2.2 Realism and platonism

Platonism is standardly defined as the view that mathematical objects exist. On this account, my sufficient conditions for being a realist are sufficient conditions for being a platonist as well.

However, it's my impression that that most people who call themselves platonists also share some other beliefs, beliefs which mere realism does not commit one to. In particular, I suspect that many of them would like to claim that *certain specific independent questions have correct answers*, or even that *all* questions phrased in purely logical and mathematical vocabulary must have correct answers.

Insofar as these platonists typically also think that we can *learn* the right answers to these questions, they may face extra epistemic burdens. But I won't attempt to show that these burdens can be met in this paper. For recall that solving the that access problem only requires that one reconcile naturalism with the (apparently obvious) claim that we have substantial knowledge of mathematics, and with a realist view of what mathematical knowledge is - *not* that one reconcile naturalism with the (much less obvious) claim that we are potentially omniscient about math, or that we are in a position to learn the right answer to some one particular question, which appears to be remote from any of the simple mathematical claims that feel obvious to us.

Note that accounting for mathematical knowledge, is all that is required to block transcendental arguments to the effect that we obviously have mathematical knowledge, so realism, (or naturalism) must be wrong. There is no corresponding transcendental argument that we must accept radical conclusions in order to account for our being mathematically omniscient, or being in a position to discover the right answer to questions about the continuum hypothesis or large cardinals. For, it's far from obvious that either of these claims is actually true.

3 Solving the Access Problem

3.1 Two Questions About Access

My realist answer to the access problem starts by slicing the access problem into two component questions:

(Q1) How do mathematicians manage to reliably accept mathematical doctrines which are (syntactically) consistent, and allow for whatever practical applications we're inclined to make of them?

(Q2) How, given that mathematicians can manage to do the above, do they wind up reliably accepting mathematical **truths**?

Separating up the problem like this helps the realist, because we should notice that (Q1) is everyone's problem. Any theory giving a philosophical interpretation of mathematical practice, needs to account for how mathematicians manage to make (syntactically) consistent claims, which don't lead us to build bridges that fall down. Thus, for the purposes of addressing the challenge that *realism in particular* is committed to a spooky pre-established harmony between the human mind and a realm of objective mathematical facts, we can assume a solution to (Q1) and focus on (Q2).

Now, (Q2) *does* seem to pose a distinctive problem for the realist. The problem isn't just that some alternatives (like fictionalism) don't take mathematical statements to be true, and hence don't *literally* have to explain how mathematicians reliably wind up with true beliefs. Any mainstream philosopher of math is going to admit that "2+2=4" has *some* property which distinguishes it from "2+2=5", and makes it assertable in the math classroom. So, while the fictionalist doesn't literally wind up having to explain how mathematicians reliably assert what's true, he does have to explain why they reliably assert what's true in the relevant fiction. Any other philosophical interpretation of the human activity we call doing mathematics will face a similar question.

Rather, the issue is, that it seems like a number of different kinds of mathematical objects might be equally coherent to posit, and equally helpful (or at least harmless) to the business of making scientific predictions. Approaches like Fictionalism face no epistemological worry about how we made the right choice in (coherent) posits. But, the realist holds that existential claims in mathematics are to be taken literally. If they are true, there must be some objects referred to, and these must be as described. So, even given that we are able to come up with some coherent mathematical practice, there appears to still be a puzzle about how we managed to choose, from among various possible coherent posits, ones which fit some abstracta that actually exist?

A popular version of realism (call it reductive realism) faces this problem in an especially acute form. Reductive realism takes there to be one specially privileged range of mathematical objects - usually, the universe of sets, V. V exists, and all these other kinds of mathematical abstracta which would be equally coherent to posit, don't exist. Thus it looks like we've got tremendously lucky in positing sets, and it seems quite mysterious how this luck can be explained.

In what follows, I will aim to show that realism is compatible with naturalism, even in this worst-case scenario.

4 Realism and past mathematicians

Now, as regards giving a suitable reductive realist explanation of (Q2), I propose that the way forward is to think about the past. What would a reductive realist say about Euler's beliefs about numbers?

Surely, they will say that most of Euler's beliefs about numbers were true, even though he didn't realize that numbers were sets. Indeed, not only were these beliefs true, but Euler had knowledge. Now, if he had said 'numbers are the most fundamental mathematical objects' or 'numbers are the only mathematical objects' then (for a reductive realist) he would have been wrong about the philosophy of math. But, this wouldn't change the fact that his derivations of true ordinary mathematical claims were sufficient to give him knowledge. Thus, (even assuming reductive realism) it would seem that you don't need to have the notion of 'set' to make statements which ultimately are about sets[4].

Now, suppose that we discover a secret cache of Arabic mathematics that talks consistently about some other hitherto unimagined kind of mathematical objects, called 'zilfs'. Just as we identify the numbers with a part of the realm of the sets, wouldn't we also identify the 'zilfs' with certain suitably chosen sets? Indeed, wouldn't we credit the authors of correct derivations about the

[4]Note that this kind of treatment is nicely parallel with with we say in the sciences. Past scientists had various true beliefs about heat transfer before they realized that heat was molecular motion, and similarly, they had true beliefs about about oxygen or gold before they learned the internal structure of these elements.

zilfs with knowledge?

One might worry that this interpretive approach is realism in name only, as it rules out the possibility of making incorrect assumptions about the integers. However, this is not the case. Had Euler made one false assumption about the numbers, we might still have interpreted him as talking about the numbers, and simply taken that assumption to be in error - without undermining his claims to have genuine knowledge of his true conclusions about the numbers. All I argue here is that, had Euler's (consistent) assumptions about the numbers been *so throughgoingly mistaken* that it would undermine his claim to knowledge, then we would have interpreted him as working in some other system in which his methods were reliable enough to yield knowledge.

Another worry is that one wouldn't be able to find a suitable interpretation of all coherent practices, as 'really' talking about the sets. We have produced interpretations of diverse mathematical-looking practices in terms of the sets, in various particular cases, but we don't have a general reason to think that there will always be one. Note that the completeness theorem does not ensure this! It only says that the hierarchy of sets contains models of all consistent *first order* theories, but some mathematical practices, like arithmetic, have no adequate first order axiomatization.

However, if we don't have a reason to think that every consistent mathematical-looking practice can be interpreted as quantifying over the *sets*, we do have reason to think that every such practice can be interpreted in terms of *the fundamental mathematical objects, whatever those are.* This is because, the whole point of taking the sets to be the fundamental mathematical objects, is that talk which appears to be about something else (numbers, functions etc.) can be satisfactorily cashed out purely in terms of the sets. When we seem to be positing objects with one structure, we are really talking about just a part, or aspect, of the structure of the fundamental objects. The existence of these attractive reductions is what makes sets look like a good candidate for being the fundamental mathematical objects - perhaps even what gave people the idea that there might be a unique kind of fundamental mathematical object in the first place. So, any case where we can't understand some consistent, mathematical-looking, practice as talking about some aspect of what we take to be the fundamental mathematical objects, would be evidence that we had made a wrong guess about what the fundamental mathematical objects are. Thus, our philosophical views about math might be wrong, but our mathematical knowledge would stand.

Now, if this kind of interpretive charity applies to mathematicians *in the past*, I propose that it should also apply to (as it were) mathematicians *in different possible worlds*, i.e., to different ways that the actual history of mathematics could have gone. If we had had posited different, consistently described, mathematical objects rather than numbers or sets, these objects too should be understood as part of the universe of sets (assuming sets are the fundamental mathematical objects). Thus, we would have been asserting truths, just as much as Euler was asserting truths about the sets when he was talking about the numbers. Conversely, even if we are wrong about sets being the fundamental mathematical object, our ordinary mathematical claims would still be right. Thus, we have a solution to the access problem.

5 Boolos and plenetudinus platonism

The theory just advocated may seem reminiscent of ontological maximalism (the view that there are mathematical objects corresponding to all consistent theories) or neo-Fregeian views, on which new kinds of abstract objects can be introduced by merely laying down sufficiently clear, consistent, 'abstraction principles' saying when two such objects are supposed to be different from one another. Now George Boolos has famously posed a problem for such views, so it's worth checking whether the same problem applies to the view I am advocating here.

Boolos points out that there can be consistent mathematical principles which are not consistent when combined [5]. His most famous example presents abstraction principles for both cardinalities and parities. While either principle is consistent on its own, the principle for cardinalities requires one consider infinite collections while the principle for parities precludes this. Thus accepting both principles leads to contradiction. This might seem to show there are two coherent practices which can't both yield literally true conclusions.

But note that anyone with a coherent practice matching the abstraction principle for parities, can't be inclined take any property to apply to infinitely many things. Thus, while one possible interpretation of their behavior would be to say that they accepted the full abstraction principle for parities (which requires that there are only finitely many objects), an (arguably superior) interpretation is available. Why not interpret those who appear to accept the abstraction principle for parities as having some kind of implicit quantifier restriction to the finite collections of objects which they are actually willing to make claims about. That is, one might think that people in this situation would not really count as accepting the abstraction principle, but rather a weaker theory, which only says that there are parities associated with *certain* properties (e.g. properties which apply to objects within *within the relevant restricted domain*).

[5] The Standard of Equality of Numbers. In George Boolos (ed.), Meaning and Method: Essays in honor of Hilary Putnam, 26178. Cambridge: Cambridge University Press. Reprinted in Boolos 1998: 20219.

6 Independent Questions

A point that is worth reiterating here, is that this account still allows us make sense of the intuitive notion that it's possible to get mathematical questions which are independent of all currently accepted axioms *wrong*. It might seem troubling that once we have assured ourselves that a potential new axiom is independent of our current axioms, the reductive platonist must admit that adding the axiom would lead us to state where most of our mathematical practice still count as giving us knowledge, and so would adding its negation! So, how can there be a right answer about whether the axiom is true?

Note that the challenge here, isn't to explain why all our mathematical views are correct (mathematicians can make mistakes), but rather to explain how our mathematical beliefs are largely correct, and our methods of forming them are reliable, to whatever extent is required for these methods to count as giving us mathematical knowledge.

So, what happens if a proposed new axiom is false, but my community comes to accept it? Well, there are two possibilities. One option is that adding the axiom makes a small difference to their resulting mathematical practice. In this case there's no change in meaning, and that particular axiom expresses a falsehood, but the result is still a community that's reliable enough to count as having substantial mathematical knowledge. Alternatively the mathematical practice might be so radically altered that, on the old interpretation, its conclusions would be largely unjustified. But in this case, the radical change in practice will also suitably change the meaning of the relevant mathematical terms, yielding a new interpretation on which the conclusions condoned by the practice would (largely) count as mathematical knowledge.

7 Conclusion

Putting all of the above together, we get an answer to the access problem which satisfies our initial characterization of realism about mathematics, and fits other common intuitions as well. As required by my initial stipulation: there are mathematical objects, mathematical claims can be true or false independent of any human activity, and we can meaningfully wonder about questions that are independent of our currently accepted axioms and formal methods of reasoning. But adopting this view also yields some other intuitive consequences: People are reliable, but not infallible about mathematics. It's easier to count as being right about math, than about philosophy of math (e.g. claims about which mathematical objects are most fundamental). And, in fact, the strength and prevalence of a mathematical judgement correlates with its reliability.

We get the last result, that mathematical judgments that feel obvious, or are widely shared, are more likely to be correct than weaker or more controversial mathematical intuitions, because the former carry more weight in our interpretive practice. That is, the correct interpretation of a mathematical practice will (ceterus paribus) take more pains to make wide-spread or obvious-feeling judgements come out as true than tentative or controversial ones. Thus, while we don't get the kind of perfect certainty regarding mathematical intuitions about, say, large cardinals, which has struck many as deeply counterintuitive, we can still justify our extreme confidence regarding less controversial mathematical claims.